Infinitesimal isometries in Riemannian 2-manifolds: Bryant's example

1 Existence of infinitesimal isometries on Riemannian 2-manifolds

Let (M, g) be a smooth Riemannian manifold of dimension n. A smooth vector field ξ on M is an infinitesimal isometry (or a Killing field) if and only if ξ satisfies

$$L_\xi g = 0, \tag{1.1}$$

where L is the Lie derivative. In terms of local coordinates $x = (x^1, \cdots, x^n)$ (1.1) becomes

$$\xi_i^\lambda g_{\lambda j} + \xi_j^\lambda g_{\lambda i} - \xi^\lambda g_{ij,\lambda} = 0, \quad i, j = 1, \cdots, n, \tag{1.2}$$

where $g_{ij} = g(\partial_i, \partial_j)$ and $\xi = \xi^\lambda \frac{\partial}{\partial x^\lambda}$ (summation convention for $\lambda = 1, \cdots, n$). Since (1.2) is symmetric in (i, j) the number of equations in (1.2) is $\frac{n(n+1)}{2}$ whereas the number of unknowns is n so that (1.2) is overdetermined if $n \geq 2$. In this section we shall present a coordinate-free computation of prolongation of (1.1) with $n = 2$ to a complete system of order 2 and discuss the existence of solutions. Let $\{e_1, e_2\}$ be an orthonormal frame over a 2-dimensional Riemannian manifold M and let ω^1, ω^2 be the dual coframe. Then

$$g = \omega^1 \circ \omega^1 + \omega^2 \circ \omega^2,$$

where $\phi \circ \eta := \frac{1}{2}(\phi \otimes \eta + \eta \otimes \phi)$ is the symmetric product of 1-forms. Recall also that there exist a uniquely determined 1-form ω_2^1 (Levi-Civita connection) and a function K (Gaussian curvature) satisfying

$$\left.\begin{aligned} d\omega^1 &= -\omega_2^1 \wedge \omega^2 \\ d\omega^2 &= \omega_2^1 \wedge \omega^1 \end{aligned}\right\} \tag{1.3}$$

and

$$d\omega_2^1 = K\omega^1 \wedge \omega^2. \tag{1.4}$$

Furthermore, Lie derivatives of ω^i, $i = 1, 2$ with respect to a vector field $\xi = \xi^1 e_1 + \xi^2 e_2$ are

$$\begin{array}{rcl} L_\xi \omega^1 & = & d(\xi \lrcorner \omega^1) + \xi \lrcorner d\omega^1 \\ & = & d\xi^1 - \omega_2^1(\xi)\omega^2 + \xi^2 \omega_2^1 \ \text{ by (1.3)} \end{array} \tag{1.5}$$

and similarly

$$\begin{array}{rcl} L_\xi \omega^2 & = & d(\xi \lrcorner \omega^2) + \xi \lrcorner d\omega^2 \\ & = & d\xi^2 + \omega_2^1(\xi)\omega^1 - \xi^1 \omega_2^1. \end{array} \tag{1.6}$$

By (1.5) and (1.6), we have

$$\begin{array}{rcl} \frac{1}{2} L_\xi g & = & (L_\xi \omega^1) \circ \omega^1 + (L_\xi \omega^2) \circ \omega^2 \\ & = & (d\xi^1 + \xi^2 \omega_2^1) \circ \omega^1 + (d\xi^2 - \xi^1 \omega_2^1) \circ \omega^2. \end{array}$$

On the other hand, the covariant derivative of ξ is a $(1, 1)$ tensor field given by

$$\begin{array}{rcl} \nabla \xi & = & \nabla(\xi^1 e_1 + \xi^2 e_2) \\ & = & (d\xi^1 + \xi^2 \omega_2^1) \oplus e_1 + (d\xi^2 - \xi^1 \omega_2^1) \oplus e_2. \end{array}$$

By setting

$$\left\{ \begin{array}{rcl} d\xi^1 + \xi^2 \omega_2^1 & = & \xi_1^1 \omega^1 + \xi_2^1 \omega^2, \\ d\xi^2 - \xi^1 \omega_2^1 & = & \xi_1^2 \omega^1 + \xi_2^2 \omega^2 \end{array} \right. \tag{1.7}$$

and substituting in the above we have

$$\frac{1}{2} L_\xi g = \xi_1^1 \omega^1 \circ \omega^1 + (\xi_2^1 + \xi_1^2)\omega^1 \circ \omega^2 + \xi_2^2 \omega^2 \otimes \omega^2.$$

By (1.1), ξ is an infinitesimal isometry if and only if

$$\xi_1^1 = \xi_2^2 = 0, \ \xi_2^1 + \xi_1^2 = 0. \tag{1.8}$$

Substituting (1.8) in (1.7) we see that a vector field $\xi = \xi^1 e_1 + \xi^2 e_2$ is an infinitesimal isometry if and only if

$$\left\{ \begin{array}{rcl} d\xi^1 & = & -\xi^2 \omega_2^1 + \xi_2^1 \omega^2, \\ d\xi^2 & = & \xi^1 \omega_2^1 + \xi_1^2 \omega^1, \end{array} \right. \tag{1.9}$$

which is a coordinate-free version of (1.2) with $n = 2$ expressed as an exterior differential system. Prolongation of (1.9) to a complete system is differentiating (1.9) and expressing $(d\xi^1, d\xi^2, d\xi^1_2)$ in terms of (ξ^1, ξ^2, ξ^1_2) :
We apply d to (1.9) and substitute (1.9), (1.3) and (1.4) for $d\xi^i$, $d\omega^i$ and $d\omega^1_2$, respectively, to obtain

$$(d\xi^1_2 - K\xi^2\omega^1) \wedge \omega^2 = 0,$$
$$(d\xi^1_2 + K\xi^1\omega^2) \wedge \omega^1 = 0.$$

Hence we have

$$d\xi^1_2 = K(\xi^2\omega^1 - \xi^1\omega^2). \tag{1.10}$$

The system (1.9) and (1.10) is a prolongation of (1.1) to a complete system. Now consider the Euclidean space $\mathbb{R}^3$ of variables (ξ^1, ξ^2, ξ^1_2). Then the submanifold of the first jet space of ξ defined by (1.8) may be identified with $\mathcal{S} := M \times \mathbb{R}^3$.
On $M \times \mathbb{R}^3$ consider the Pfaffian system $\theta = (\theta^1, \theta^2, \theta^3)$ given by

$$\begin{aligned} \theta^1 &= d\xi^1 + \xi^2\omega^1_2 - \xi^1_2\omega^2, \\ \theta^2 &= d\xi^2 - \xi^1\omega^1_2 + \xi^1_2\omega^1, \\ \theta^3 &= d\xi^1_2 - K\xi^2\omega^1 + K\xi^1\omega^2. \end{aligned} \tag{1.11}$$

We check the Frobenius integrability conditions for (1.11): By (1.3) and (1.4) we have

$$d\theta^1, d\theta^2 \equiv 0 \mod \theta$$

and

$$d\theta^3 \equiv (K_1\xi^1 + K_2\xi^2)\omega^1 \wedge \omega^2 \mod \theta$$

where $K_i = dK(e_i)$, $i = 1, 2$ so that $dK = K_1\omega^1 + K_2\omega^2$.
Thus (1.11) is integrable if and only if $T := K_1\xi^1 + K_2\xi^2$ is identically zero on $M \times \mathbb{R}^3$, which is equivalent to $K_1 = K_2 = 0$ i.e. K is constant. In this case, there exist 3 parameter family of solutions by the Frobenius theorem. Otherwise, assuming $dT \neq 0$ on $T = 0$, we consider a submanifold $\mathcal{S}'$ of dimension 4 defined by $T = 0$.
Differentiating $dK = K_1\omega^1 + K_2\omega^2$, we see by (1.3) that

$$\begin{aligned} 0 &= d^2K \\ &= (dK_1 + K_2\omega^1_2)\omega^1 + (dK_2 - K_1\omega^1_2)\omega^2. \end{aligned} \tag{1.12}$$

Thus we put

$$dK_1 = -K_2\omega_2^1 + K_{11}\omega^1 + K_{12}\omega^2, \quad (1.13)$$
$$dK_2 = K_1\omega_2^1 + K_{21}\omega^1 + K_{22}\omega^2. \quad (1.14)$$

By substituting (1.13), (1.14) in (1.12) we have $K_{12} = K_{21}$.

On $\mathcal{S}'$, we have by (1.11), (1.13) and (1.14)

$$\begin{aligned} dT &= \xi^1 dK_1 + K_1 d\xi^1 + \xi^2 dK_2 + K_2 d\xi^2 \\ &\equiv (K_{11}\xi^1 + K_{12}\xi^2 - K_2\xi_2^1)\omega^1 + (K_{12}\xi^1 + K_{22}\xi^2 + K_1\xi_2^1)\omega^2 \quad \mod \theta. \end{aligned}$$

We set

$$\begin{cases} T_1 = K_{11}\xi^1 + K_{12}\xi^2 - K_2\xi_2^1, \\ T_2 = K_{12}\xi^1 + K_{22}\xi^2 + K_1\xi_2^1. \end{cases} \quad (1.15)$$

If $T_1, T_2 \equiv 0$ on $\mathcal{S}'$, $i^*\theta^1, i^*\theta^3, i^*\theta^3$ have rank 2 by Theorem **??**. Then $\mathcal{S}'$ is foliated by two dimensional integral manifolds and therefore there are 2 parameter family of solutions. But this implies that $K_1 = K_2 = 0$ which is impossible.

Let $A = \begin{pmatrix} K_1 & K_2 & 0 \\ K_{11} & K_{12} & -K_2 \\ K_{12} & K_{22} & K_1 \end{pmatrix}$.

If $\det A = 0$, A has rank 2 and $\mathcal{S}'' = \{T = T_1 = T_2 = 0\}$ is a 3-dimensional submanifold of $\mathcal{S}$. If we have $dT_1, dT_2 \equiv 0 \mod \theta^1, \theta^2, \theta^3$ on $\mathcal{S}''$, Theorem **??** and the Frobenius theorem imply that $\mathcal{S}''$ is foliated by two dimensional integral manifolds and therefore there exists 1 parameter family of solutions. To calculate dT_1, dT_2 we differentiate (1.13). Then we have

$$\begin{aligned} 0 &= d^2K_1 \\ &= (dK_{11} + 2K_{12}\omega_2^1 + K_2K\omega^2)\omega^1 + (dK_{12} + K_{22}\omega_2^1 - K_{11}\omega_2^1)\omega^2. \end{aligned} \quad (1.16)$$

Thus we put

$$dK_{11} = -2K_{12}\omega_2^1 + K_{111}\omega^1 + K_{112}\omega^2, \quad (1.17)$$
$$dK_{12} = (K_{11} - K_{22})\omega_2^1 + K_{121}\omega^1 + K_{122}\omega^2. \quad (1.18)$$

By substituting (1.17), (1.18) in (1.16) we have $K_{112} = K_{121} - K_2K$.

Differentiating (1.14), we have

$$\begin{aligned} 0 &= d^2K_2 \\ &= (dK_{12} + K_{22}\omega_2^1 - K_{11}\omega_2^1)\omega^1 + (dK_{22} - 2K_{12}\omega_2^1 + K_1K\omega^1)\omega^2. \end{aligned} \tag{1.19}$$

By substituting (1.17), (1.18) in (1.19) we have

$$(dK_{22} - 2K_{12}\omega_2^1 + K_1K\omega^1 - K_{122}\omega^1)\omega^2 = 0.$$

Thus we put

$$dK_{22} = 2K_{12}\omega_2^1 + (K_{122} - K_1K)\omega^1 + K_{222}\omega^2. \tag{1.20}$$

On $\mathcal{S}''$, we have by (1.11), (1.17), (1.18) and (1.20)

$$\begin{aligned} dT_1 \equiv\ & (K_{111}\xi^1 + (K_{121} - K_2K)\xi^2 - 2K_{12}\xi_2^1)\omega^1 \\ & +(K_{121}\xi^1 + K_{122}\xi^2 + (K_{11} - K_{22})\xi_2^1)\omega^2 \quad \mod \theta \end{aligned}$$

and

$$\begin{aligned} dT_2 \equiv\ & (K_{121}\xi^1 + K_{122}\xi^2 + (K_{11} - K_{22})\xi_2^1)\omega^1 \\ & +((K_{122} - K_1K)\xi^1 + K_{222}\xi^2 + 2K_{12}\xi_2^1)\omega^2 \quad \mod \theta. \end{aligned}$$

We summarize the discussions of this section in the following

Theorem 1.1 *Let M be a Riemannian manifold of dimension 2.*

Let $\mathbf{K} = \begin{pmatrix} K_1 & K_2 & 0 \\ K_{11} & K_{12} & -K_2 \\ K_{12} & K_{22} & K_1 \\ K_{111} & K_{121} - K_2K & -2K_{12} \\ K_{121} & K_{122} & K_{11} - K_{22} \\ K_{122} - K_1K & K_{222} & 2K_{12} \end{pmatrix}.$

- *(i) If $\mathbf{K}$ has rank 0, there exist 3 parameter family of infinitesimal isometries,*
- *(ii) If $\mathbf{K}$ has rank 2 and $(K_1, K_2) \neq 0$, there exist 1 parameter family of infinitesimal isometries,*
- *(iii) If $\mathbf{K}$ has rank 3, there exists only trivial infinitesimal isometry.*